

## A Comparative Study on the Power of van Lieshout and Baddeley's J-Function

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### *Summary*

Summary functions such as the empty space function  $F$  and the nearest neighbour distance distribution function  $G$  are often used as test statistics for point patterns. Van Lieshout and Baddeley recently proposed an alternative statistic, the  $J$ -function, which is defined as  $J = (1 - G)/(1 - F)$ . Theoretical advantages of the  $J$ -function over the  $F$ - and  $G$ -statistics are that it measures the type, strength and range of interaction, and that it can be evaluated explicitly for a larger class of models. In this simulation study we investigate empirically how the power of tests based on  $J$  compares to that of tests based on  $F$  and  $G$ .

*Key words:* Area-interaction process; Cluster process; Empty space function; Hard core process; J-function; Monte Carlo test; Nearest neighbour distance distribution function; Power.

### *Zusammenfassung:*

Zusammenfassungsfunktionen wie die sphärische Kontaktverteilungsfunktion  $F$  und die Verteilungsfunktion des Abstandes zum nächsten Nachbarn  $G$  werden häufig benutzt, um Punktfelder auf komplette räumliche Zufälligkeit zu testen. Van Lieshout und Baddeley haben eine alternative Zusammenfassungsfunktion vorgeschlagen: die  $J$ -Funktion, die als  $J = (1 - G)/(1 - F)$  definiert ist. Die  $J$ -Funktion mißt nicht nur die Art, Stärke und das Ausmaß der Interaktion, sondern kann auch für eine größere Anzahl von Modellen als die  $F$ - oder die  $G$ -Funktion analytisch berechnet werden. In dieser Simulationsstudie untersuchen wir die Güte von Tests, die auf  $J$  basieren, im Vergleich zu Tests, die auf  $F$  oder  $G$  basieren.

## 1. Introduction

Mapped spatial patterns arise in a variety of contexts, ranging from the positions of cell nuclei in cytology, trees in a forest, nest locations of birds or other animals to the findings of ore in material science.

The statistical analysis of such a mapped point pattern usually begins with a test for spatial randomness [DIGGLE, 1983; RIPLEY, 1981]. As a test statistic, low-dimensional summary functions such as the empty space function  $F$ , the nearest neighbour distance distribution function  $G$  or the reduced second moment function  $K$  are used (cf. [DIGGLE, 1983; RIPLEY, 1981, STOYAN, KENDALL, and MECKE, 1995]). Roughly speaking the  $K$ -function is proportional to the expected number of other events, that is points of the pattern, in balls of various radii around a typical event. Both  $F$  and  $G$  are distribution functions of distances between points: the empty space function  $F$  of the distance from an arbitrary sampling point, for example the origin, to the nearest event and  $G$  of the distance from an event to the nearest other event. Recently, VAN LIESHOUT and BADDELEY (1996) proposed an alternative, the  $J$ -function, which compares inter-event distances to distances from a fixed sampling point.

In this paper we describe a simulation study which compares the power of tests for complete spatial randomness based on the  $J$ -function with the power of tests based on  $F$  and  $G$ . We will start by reviewing the theoretical background of  $F$  and  $G$  and use this to introduce the  $J$ -function in Section 2. Section 3 presents the models we use as alternatives to a random scattering of points and Section 4 describes the simulation study. The last section discusses the results thus obtained.

## 2. Theoretical Background

Throughout this paper we assume that  $X$  is a stationary and isotropic point process in  $\mathbb{R}^2$ . The empty space function  $F$  of  $X$  is then defined as

$$F(r) = \mathbb{P}(d(0, X) \leq r) \quad r \geq 0 \quad (1)$$

where  $d(0, X) = \min \{\|x\| : x \in X\}$ , the distance from the origin to the nearest event of  $X$ . Similarly,

$$G(r) = \mathbb{P}^{10}(d(0, X) \leq r) \quad (2)$$

where  $\mathbb{P}^{10}$  denotes the reduced Palm distribution of  $X$ . Heuristically this is the conditional distribution of the remainder  $X \setminus \{0\}$  of  $X$  given there is an event at the origin. Further details can be found in [DIGGLE, 1983; STOYAN et al., 1995].

In [LIESHOUT and BADDELEY, 1996], the  $J$ -function is defined as the ratio

$$J(r) = \frac{1 - G(r)}{1 - F(r)} \quad \text{for all } r \geq 0 \text{ such that } F(r) < 1. \quad (3)$$

It can be interpreted as the ratio of two survival functions, namely of the distance to the nearest event from (a) an event or (b) an arbitrary sampling point. Thus (3)

compares the environment of a typical event of  $X$  to the environment of an arbitrary sampling point.

If  $J(r) < 1$ , then the survival function of (a) is smaller than that of (b) indicating clustering, whereas if  $J(r) > 1$ , then the survival function of (b) is smaller than that of (a) indicating regularity. For a Poisson point process,  $F$  and  $G$  are identical and hence  $J(\cdot) \equiv 1$ . Thus, a single plot of the (estimated)  $J$ -function provides valuable information on the type and strength of interaction between events. Furthermore, it can be shown that, in contrast to  $F$  and  $G$ , the  $J$ -function is constant beyond the effective range of interaction.

However, the price to pay for the lowering of dimension necessary to be able to visualise a summary statistic graphically is that neither  $J$  nor  $F$ ,  $G$  or  $K$  completely determine the distribution of  $X$ ; [BEDFORD and BERG, 1997] gives an example of a point process that is not Poisson but for which  $J(\cdot) \equiv 1$ .

The  $J$ -function (3) can be expressed in terms of the Papangelou conditional intensity [KALLENBERG, 1984] (provided it exists) which is often easier to work with than the distribution of  $X$ . Thus  $J$  can be evaluated for a wider range of point process models than  $F$  and  $G$ . It also behaves well under independent superposition of point patterns, a property that is very useful when studying interactions in mapped patterns consisting of different types of events.

For formal proofs and further details see [LIESHOUT and BADDELEY, 1996]. Multivariate extensions of (3) are studied in [LIESHOUT and BADDELEY, 1999]. In [BADDELEY et al., 1997] the robustness of  $J$  against edge effects caused by incomplete observation of  $X$  is examined.

Clearly the  $J$ -function has many theoretical advantages over  $F$  and  $G$ , but how does it perform empirically? In this paper we will give some answers to this question by means of simulation.

### 3. The Simulated Processes

In order to compare the power of tests based on  $J$  with tests based on  $F$  and  $G$  (see (1)–(3)), we simulated three different types of non-Poissonian point processes: cluster processes, hard core processes and area-interaction processes. We then tested the simulated patterns against a Poisson null hypothesis and estimated the power of these tests by the proportion of rejected patterns. In this section we will describe the simulated processes; the testing procedures used will be explained in Section 4.

For all of the following processes we obtained samples on a unit square window.

#### 3.1 Cluster process

A cluster process is derived from a parent process by scattering a cluster of daughter points around each of the parent points. The union of all daughters then forms the

cluster process. Both the random number of daughter points in each cluster as well as the locations of the daughter points relative to the parent point are independent and identically distributed. We chose a Poisson cluster process for which the parent process is a stationary Poisson process. To each parent point we assigned a number of daughter points which is a Poisson random variable of mean  $\mu$ . The location of the daughter points is radially normal, centred at the parent point and with standard deviation  $\sigma$ . The simulation of a Poisson cluster process is straightforward; see [STOYAN and STOYAN, 1944] for a description. We assumed periodic boundary conditions.

### 3.2 *Hard core process*

In a hard core process all events are a minimum distance, the hard core distance  $h$ , apart from each other. We chose a Type 2 Matérn hard core process using the algorithm described in [RIPLEY, 1977]. This process leads to regular patterns with the strength of repulsion depending on the hard core distance. We simulated the point pattern on a window of size  $1 + 2h$  and then only sampled the points with in the central unit square, thus avoiding edge-effects.

### 3.3 *Area-interaction process*

This process was first introduced as the penetrable spheres mixture model in [WINDOM and ROWLINSON, 1970] and then extended in [BADDELEY and LIESHOUT, 1995]. It is defined by the density

$$p(\underline{x}) = \alpha \beta^{n(\underline{x})} \gamma^{-|\underline{x} \oplus K|} \quad (4)$$

with respect to a unit rate Poisson point process on a compact window. Here  $\alpha$  is the normalizing constant and  $n(\underline{x})$  is the number of events in  $\underline{x}$ . The parameter  $\beta > 0$  influences the intensity of the process. The parameter  $\gamma$  does not only control the strength but also the type of interaction. For  $\gamma < 1$ , realisations of (4) tend to be regular, whereas for  $\gamma > 1$  clustered patterns are more likely. The strength of interaction depends on the area  $|\underline{x} \oplus K|$  covered by the union of congruent compact sets  $K$  centred at the events in  $\underline{x}$  (using the symbol  $\oplus$  for the Minkowski addition, i.e.  $\underline{x} \oplus K = \{x + k \mid x \in \underline{x}, k \in K\}$ ). In our simulations, we chose a square for  $K$ .

The normalizing constant  $\alpha$  in (4) is not computable in closed form, so we have to use a Markov Chain Monte Carlo (MCMC) method to produce samples. We used the algorithm outlined in [KENDALL, 1998], coded in his C program Perfect. This algorithm uses the coupling from the past idea of PROPP and WILSON [1996] which, in contrast to conventional MCMC methods, allows for exact rather than approximate samples. The algorithm assumes empty boundary conditions. For area-interaction processes with  $\gamma > 1$ , simpler exact sampling algorithms have been developed, see [HÄGGSTRÖM, LIESHOUT, and MØLLER, 1999; THÖNNES, 1999].

## 4. Testing procedure

### 4.1 Estimation of summary statistics

As we saw in Section 2,  $F$  and  $G$  are defined in terms of distances between points. However, since in practice  $X$  is observed within a bounded window  $W$  only, inter-point distances based on  $X \cap W$  for points close to the border may well differ from the 'true' distances. To deal with these edge effects, we map the point pattern onto a torus and regard the observation window as the centre of a  $9 \times 9$  grid of windows with identical point patterns. The events in the other windows are taken into account when determining the nearest neighbour distances in the centre window. Note that the torus condition is only one of many possible ways to deal with edge effects, but in general it performs well, although it sometimes loses power for regular point patterns [RIPLEY, 1979]. Surprisingly, tests based on the  $J$ -function do not become less powerful if naive estimators which are not corrected for edge effects are used, see [BADDELEY et al. 1997].

We estimate the summary statistics (1)–(3) as follows. For each event we determine the distance to its nearest neighbour and use the empirical cumulative distribution function of these distances as an estimator for  $G$ . The estimation of  $F$  requires a set of sampling points, which can be a regular grid or a random set of points. We followed the recommendation in [DIGGLE, 1979] and used a regular grid of sampling points with mesh size 0.1. For each of these sampling points the distance to the nearest event is determined and the empirical distribution function of these distances yields an estimator for  $F$ . Finally, the  $J$ -function is simply estimated by a ratio estimate.

### 4.2 Test statistics

In order to define a test statistic, a measure of discrepancy between the estimated and theoretical null hypothesis values of a given summary function has to be chosen. In the simulation study below we consider a null hypothesis of complete spatial randomness (CSR). Under this hypothesis, the point pattern is produced by a stationary Poisson process of intensity  $\lambda > 0$ . It is easily seen that under CSR, the empty space function and the nearest neighbour distance distribution coincide and are given by

$$F_0(r) = G_0(r) = 1 - \exp(-\lambda\pi r^2).$$

Hence  $J_0(r) = 1$  (cf. Section 2).

In the following let  $H_0$  be the theoretical summary function under CSR and  $\hat{H}_1$  the estimate of the appropriate summary function for the observed point pattern. If some prior knowledge about the process to be tested is available, for example its interaction radius, then it might be sufficient to use a statistic which is based on

the difference between the theoretical and estimated summary function at only one specific value of  $r$ :

$$u_1(r) = |\hat{H}_1(r) - H_0(r)|.$$

However, it is usually more sensible to evaluate more information given by the point pattern and compare values over a range of  $r$ . More specifically, we consider the following two test statistics.

### 1. Maximum Statistic

$$u_1 = \max_{0 \leq r \leq S} |\hat{H}_1(r) - H_0(r)|$$

### 2. Integral Statistic

$$u_1 = \int_0^S (\hat{H}_1(r) - H_0(r))^2 dr.$$

Here  $S$  denotes the upper limit to the range of  $r$ -values considered.

Hardly any distributional theory for the test statistics  $u_1$  seems available, so we have to resort to Monte Carlo tests. Thus, the value of  $u_1$  for the data pattern is compared with values  $u_2, \dots, u_m$  obtained from  $m - 1$  independent simulations of the null hypothesis. The rank of  $u_1$  then yields an exact test, since under the null hypothesis  $\mathbb{P}(u_1 = u_{(j)}) = \frac{1}{m}$ , where  $u_{(j)}$ ,  $j \in \{1, \dots, m\}$  denotes the  $j$ th order statistic. As DIGGLE (1979) points out  $m = 100$  is usually sufficient since for greater  $m$  the power of the test increases only marginally with  $m$ .

## 5. Results

In the simulation study below, a Poisson null hypothesis was tested against the three alternative models discussed in Section 3. We varied the parameter settings and thus the degree of interaction for each model, tuning the overall intensity to 50 points per unit area. For each parameter setting we simulated 100 point patterns and performed the Monte Carlo tests. The number of rejected patterns yields an estimate for the power of the test. The relative frequencies of rejection of the null-hypothesis are summarized in Tables 1–10 and in the following are interpreted as an estimate for the power of the tests for CSR.

Monte Carlo tests were performed using the Maximum and Integral Statistics (cf. Section 4) at a significance level of 5%. Thus 99 realisations of a Poisson point process with intensity  $\lambda = 50$  were generated and the null hypothesis was rejected if the rank of  $u_1$  was 96 or above. Regarding the range of  $r$ -values, recall that  $J(r)$  is only defined for  $r$  such that  $F(r) < 1$ . Simulations suggest that for our sampling scheme the probability that the estimate of  $F$  becomes less than 1 is sufficiently low for values of  $r$  up to 0.12 and we therefore estimated the summary functions up to range 0.12.

## 5.1 Cluster process

First consider the Poisson cluster process described in Section 3.1 with parent intensity  $50/\mu$ . The degree of interaction depends on the mean number  $\mu$  of daughter points and the standard deviation  $\sigma$  of the normal location distribution. The more daughter points or the smaller the standard deviation  $\sigma$ , the more clustered the point pattern will be. This is reflected in the results summarised in Tables 1–4, where the estimated power of all tests decreases with increasing stand-

Table 1  
Estimated power for a cluster process with  $\mu = 2, \sigma = 0.06$

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.05	0.14	0.13	0.06	0.13	0.12
0.02	0.05	0.18	0.11	0.04	0.15	0.10
0.03	0.07	0.20	0.13	0.07	0.18	0.14
0.04	0.06	0.31	0.24	0.08	0.27	0.16
0.05	0.11	0.29	0.22	0.08	0.30	0.22
0.06	0.13	0.32	0.24	0.15	0.31	0.28
0.07	0.21	0.27	0.14	0.17	0.35	0.31
0.08	0.22	0.27	0.12	0.18	0.35	0.25
0.09	0.25	0.23	0.03	0.21	0.30	0.18
0.10	0.27	0.22	0.00	0.26	0.28	0.13
0.11	0.31	0.22	0.00	0.28	0.28	0.05
0.12	0.32	0.22	0.00	0.31	0.28	0.00

Table 2  
Estimated power for a cluster process with  $\mu = 2, \sigma = 0.04$

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.04	0.26	0.21	0.07	0.28	0.21
0.02	0.10	0.49	0.40	0.08	0.40	0.41
0.03	0.07	0.73	0.64	0.08	0.65	0.61
0.04	0.10	0.77	0.68	0.08	0.76	0.73
0.05	0.16	0.75	0.74	0.11	0.79	0.80
0.06	0.20	0.75	0.73	0.20	0.76	0.84
0.07	0.31	0.69	0.62	0.22	0.78	0.81
0.08	0.36	0.67	0.40	0.26	0.77	0.80
0.09	0.41	0.64	0.11	0.36	0.73	0.72
0.10	0.43	0.63	0.01	0.37	0.73	0.63
0.11	0.44	0.63	0.00	0.42	0.73	0.30
0.12	0.45	0.63	0.00	0.45	0.73	0.01

ard deviation  $\sigma$  or decreasing mean cluster size  $\mu$ . Non-monotonicities with respect to the increasing upper limit  $S$  in the Tables 1–4 are due to random fluctuations.

Overall we find that for patterns with weak clustering tests based on  $F$  are more powerful at large values of  $S$  than tests based on  $G$  and  $J$ , whereas for stronger clustered patterns tests based on  $G$  are most powerful. We expect tests based on  $G$  to be more powerful for strongly clustered patterns for the following reason. If the cluster members lie very close to each other then the distances from an event to the nearest sampling point will be approximately the same for all members of the

Table 3

Estimated power for a cluster process with  $\mu = 5, \sigma = 0.06$ 

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.04	0.33	0.29	0.03	0.25	0.25
0.02	0.05	0.55	0.47	0.05	0.47	0.45
0.03	0.06	0.68	0.64	0.09	0.63	0.60
0.04	0.23	0.77	0.76	0.13	0.73	0.76
0.05	0.31	0.76	0.81	0.25	0.77	0.81
0.06	0.45	0.73	0.81	0.37	0.80	0.84
0.07	0.55	0.76	0.84	0.45	0.81	0.91
0.08	0.60	0.76	0.57	0.55	0.84	0.93
0.09	0.64	0.74	0.27	0.59	0.80	0.91
0.10	0.71	0.74	0.01	0.62	0.79	0.80
0.11	0.74	0.74	0.00	0.66	0.78	0.51
0.12	0.74	0.74	0.00	0.71	0.77	0.08

Table 4

Estimated power for a cluster process with  $\mu = 5, \sigma = 0.04$ 

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.03	0.66	0.58	0.08	0.62	0.55
0.02	0.01	0.91	0.87	0.03	0.82	0.82
0.03	0.09	0.98	0.99	0.08	0.96	0.96
0.04	0.30	0.99	0.98	0.18	0.99	0.99
0.05	0.49	0.99	0.99	0.37	0.99	1.00
0.06	0.61	0.99	1.00	0.50	0.99	1.00
0.07	0.73	0.98	1.00	0.67	1.00	1.00
0.08	0.75	0.98	0.90	0.71	1.00	1.00
0.09	0.80	0.98	0.48	0.73	0.99	1.00
0.10	0.83	0.98	0.02	0.77	0.99	1.00
0.11	0.84	0.98	0.00	0.80	0.99	0.90
0.12	0.85	0.98	0.00	0.83	0.97	0.33



same cluster. Thus the estimated  $F$ -function will be close to the  $F$ -function of a Poisson process with intensity  $50/\mu$ . Most nearest event-event distances on the other hand will correspond to the distance to the nearest member of the same cluster. Hence  $G$  will deviate strongly from the  $G$ -function under CSR and we expect this deviation to be stronger than the corresponding deviation of  $F$ .

The  $J$ -function performs similar to the more powerful alternative among  $F$  and  $G$ . For  $\mu = 5$  it is even the most powerful test statistic for most values of the upper bound  $S$ . The power of tests based on  $J$  reduces considerably for large values of  $S$ . This is due to the fact that the variance of the estimator for  $J(r)$  increases drastically for large values of  $r$  because the estimated  $F(r)$  approaches 1. This is more noticeable for the Maximum Statistic than for the Integral Statistic as the latter is more robust and hence less affected by random fluctuations.

## 5.2 Hard core process

Our second results concern the Type 2 Matérn hard core process introduced in Section 3.2. For this model, the greater the hard core distance  $h$ , the more regular are the patterns produced by the process.

The results are summarised in Tables 5–6. It can be seen that when  $h$  increases, the power of all tests considered increases as well. For the hard core process, generally tests based on the  $F$ -function are least powerful. Tests based on  $G$  and  $J$  are similar in power; the  $J$ -function is slightly more powerful, in particular if the Integral statistic is used.

As in the case of cluster process alternatives, some decrease in power is noticeable for  $J$ -based tests when  $S$  gets large, but the effect is much less pronounced. As mentioned before the loss of power is due to the increasing variance of the esti-

Table 5  
Estimated power for a hard core process with  $h = 0.04$

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.05	0.00	0.01	0.04	0.00	0.00
0.02	0.04	0.00	0.04	0.02	0.00	0.04
0.03	0.03	0.06	0.30	0.04	0.00	0.17
0.04	0.05	0.83	0.80	0.03	0.64	0.57
0.05	0.04	0.81	0.69	0.05	0.81	0.72
0.06	0.05	0.60	0.50	0.05	0.65	0.68
0.07	0.02	0.40	0.36	0.03	0.49	0.54
0.08	0.02	0.34	0.30	0.03	0.42	0.49
0.09	0.01	0.31	0.24	0.02	0.33	0.35
0.10	0.01	0.30	0.17	0.03	0.29	0.28
0.11	0.01	0.29	0.16	0.03	0.27	0.25
0.12	0.01	0.29	0.10	0.02	0.24	0.19

Table 6  
Estimated power for a hard core process with  $h = 0.06$

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.08	0.00	0.03	0.09	0.00	0.00
0.02	0.04	0.00	0.06	0.05	0.00	0.00
0.03	0.03	0.06	0.27	0.05	0.00	0.20
0.04	0.07	0.80	0.79	0.05	0.61	0.59
0.05	0.14	0.89	0.99	0.08	0.88	0.96
0.06	0.13	0.90	1.00	0.11	0.90	1.00
0.07	0.19	0.92	1.00	0.14	0.90	1.00
0.08	0.21	0.91	0.96	0.15	0.90	0.99
0.09	0.19	0.91	0.89	0.19	0.90	0.97
0.10	0.19	0.91	0.74	0.17	0.91	0.92
0.11	0.19	0.91	0.67	0.16	0.91	0.87
0.12	0.19	0.91	0.42	0.13	0.91	0.72

mate of  $J$  both for the tested pattern as for the Poisson point patterns. The variance for the estimate of  $J$  is bounded for clustered processes as the estimate will take values between 0 and 1. In contrast there is no such bound for Poisson point patterns or hard core patterns. Thus due to averaging effect the decrease in power when testing hardcore patterns is smaller than when testing cluster processes.

### 5.3 Area-interaction process

Two parameters influence the strength of the interaction in an area-interaction process, the parameter  $\gamma$  and  $k$ , the sidelength of the square  $K$ . If  $g$  denotes the logarithm of  $\gamma$ , then for  $g < 0$  the model exhibits repulsion, whereas  $g > 0$  will lead to aggregation.

The density of the area-interaction process weights a Poisson process according to an exponential of the area of  $\underline{x} \oplus K$ , which is a functional of the empty space. Thus it seems surprising that tests based on the empty space function  $F$  are the least powerful, both in the repulsive as in the attractive case. However if we increase (respectively decrease) the area of  $\underline{x} \oplus K$ , the distance between the points of  $\underline{x}$  will increase (decrease) overproportionally, which provides the intuition why tests based on  $G$  are more powerful for area-interaction processes. The power of tests based on  $J$  is similar to that of tests based on  $G$ , see Tables 7–10.

#### 5.3.1 Repulsive area-interaction

From Tables 7 and 8, in the repulsive case the power of tests based on  $J$  is slightly higher than that of tests based on  $G$ .

Table 7

Estimated power for an area-interaction process with  $g = -200$ ,  $k = 0.1$ 

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.03	0.00	0.00	0.04	0.01	0.00
0.02	0.04	0.00	0.01	0.03	0.00	0.00
0.03	0.06	0.03	0.07	0.06	0.00	0.03
0.04	0.07	0.24	0.24	0.06	0.20	0.12
0.05	0.06	0.38	0.38	0.06	0.31	0.29
0.06	0.04	0.40	0.53	0.07	0.43	0.50
0.07	0.09	0.50	0.59	0.03	0.51	0.66
0.08	0.08	0.52	0.61	0.05	0.57	0.65
0.09	0.06	0.51	0.59	0.06	0.55	0.69
0.10	0.06	0.50	0.56	0.07	0.55	0.66
0.11	0.06	0.50	0.47	0.07	0.53	0.61
0.12	0.06	0.50	0.32	0.06	0.50	0.49

Table 8

Estimated power for an area-interaction process with  $g = -300$ ,  $k = 0.1$ 

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.02	0.00	0.01	0.03	0.00	0.01
0.02	0.05	0.00	0.03	0.05	0.00	0.02
0.03	0.04	0.03	0.12	0.04	0.00	0.05
0.04	0.06	0.47	0.25	0.05	0.37	0.21
0.05	0.06	0.59	0.59	0.04	0.57	0.41
0.06	0.06	0.74	0.68	0.03	0.72	0.66
0.07	0.10	0.78	0.75	0.04	0.80	0.78
0.08	0.11	0.79	0.77	0.04	0.81	0.82
0.09	0.07	0.78	0.76	0.06	0.81	0.82
0.10	0.05	0.77	0.71	0.06	0.82	0.80
0.11	0.05	0.77	0.59	0.06	0.82	0.77
0.12	0.05	0.76	0.45	0.04	0.81	0.67

### 5.3.2 Attractive area-interaction

In the attractive case the power of tests based on  $J$  is similar to the power of tests based on  $G$ , see Tables 9 and 10. However, as for cluster processes, the power of  $J$ -based tests reduces considerably if the upper limit  $S$  of the Integral and Maximum Statistics is increased.

Table 9

Estimated power for an area-interaction process with  $g = 180$ ,  $k = 0.1$ 

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.04	0.28	0.27	0.04	0.28	0.24
0.02	0.04	0.46	0.38	0.05	0.42	0.33
0.03	0.05	0.65	0.53	0.06	0.60	0.52
0.04	0.04	0.66	0.56	0.05	0.66	0.61
0.05	0.04	0.61	0.53	0.06	0.69	0.64
0.06	0.13	0.58	0.53	0.06	0.66	0.68
0.07	0.20	0.56	0.34	0.12	0.65	0.67
0.08	0.26	0.54	0.15	0.18	0.60	0.56
0.09	0.24	0.46	0.03	0.20	0.57	0.40
0.10	0.26	0.46	0.00	0.26	0.53	0.28
0.11	0.29	0.46	0.00	0.28	0.51	0.07
0.12	0.30	0.46	0.01	0.30	0.50	0.02

Table 10

Estimated power for an area-interaction process with  $g = 250$ ,  $k = 0.1$ 

S	Maximum Statistic			Integral Statistic		
	F	G	J	F	G	J
0.01	0.05	0.38	0.37	0.08	0.39	0.32
0.02	0.07	0.70	0.66	0.07	0.67	0.60
0.03	0.08	0.82	0.78	0.09	0.78	0.72
0.04	0.15	0.87	0.91	0.10	0.86	0.84
0.05	0.19	0.90	0.90	0.15	0.90	0.90
0.06	0.25	0.92	0.91	0.16	0.93	0.95
0.07	0.36	0.88	0.82	0.28	0.91	0.94
0.08	0.44	0.87	0.58	0.34	0.92	0.94
0.09	0.47	0.85	0.17	0.41	0.87	0.89
0.10	0.51	0.85	0.00	0.44	0.85	0.74
0.11	0.54	0.85	0.00	0.49	0.84	0.43
0.12	0.56	0.84	0.00	0.55	0.83	0.09

## 6. Conclusions

As discussed in Section 2,  $J$  has many theoretical advantages. It measures both strength and range of interaction and can be evaluated in closed form for more models than  $F$  and  $G$ . Furthermore, in contrast to  $F$  and  $G$ , a plot of the estimated  $J$ -function can be interpreted without comparing it to the plot of the summary function for a Poisson point process.

Overall we found that using the  $J$ -function to test for CSR is a competitive alternative to  $G$  and  $F$ . The  $J$ -function produces tests which are of similar power as the more powerful of the alternatives  $F$  and  $G$ . The highest power estimate is usually obtained close to the interaction radius of the underlying point process. For repulsive processes, tests based on  $J$  are often slightly more powerful than tests based on the alternative summary functions. The power of the tests often slightly decreases if too large a range of values is taken into account, that is if the upper limit  $S$  is too high. For  $G$  this effect is stronger for attractive patterns, for  $F$  it is stronger for regular patterns. This is due to the fact that  $G$  is larger for a clustered point pattern than for a Poisson point process and similarly,  $F$  is larger for regular patterns than for Poisson point processes. For tests based on  $J$ , however, the performance worsens drastically as the range of values taken into account grows, especially when using the Maximum Statistic. This is due to the fact that the  $J$ -function is a ratio and the variance of its estimator increases as the denominator approaches zero. The loss of power is more pronounced for processes with attractive interaction. Because of these observations, we would recommend to consider only those ranges for which the empty space function is sufficiently below 1 and to use the Integral Statistic, which is less affected by sampling fluctuations.

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